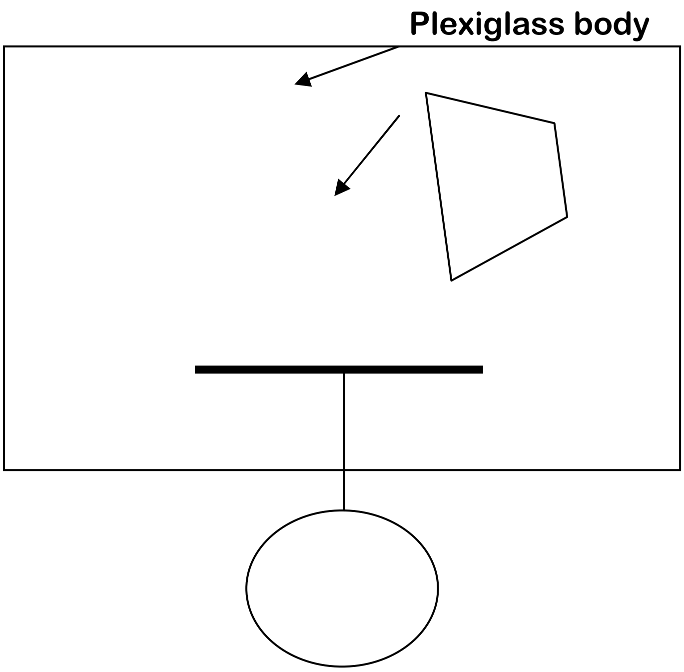
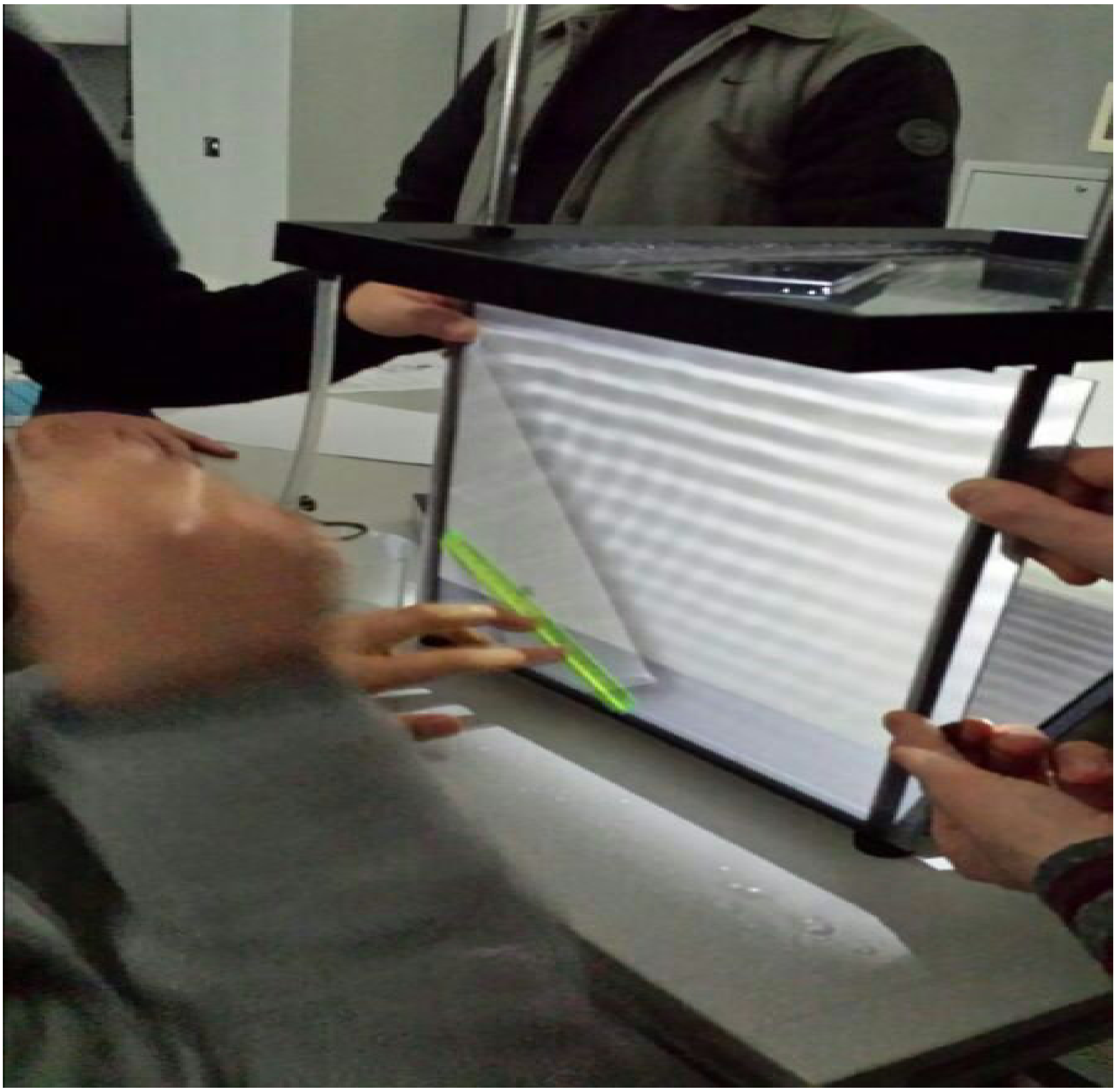
Experience with the ondoscope about light refraction.

By professor Giuseppe Caputo.

In collaboration with Antonio La Marca, IV D.
Translation by Riccardo Papa, IV D.

The refraction is that physical phenomenon consisting of a wave that changes its speed and its direction of diffusion, always perpendicular to the wave front, by passing through the separation surface between two mediums with different densities.





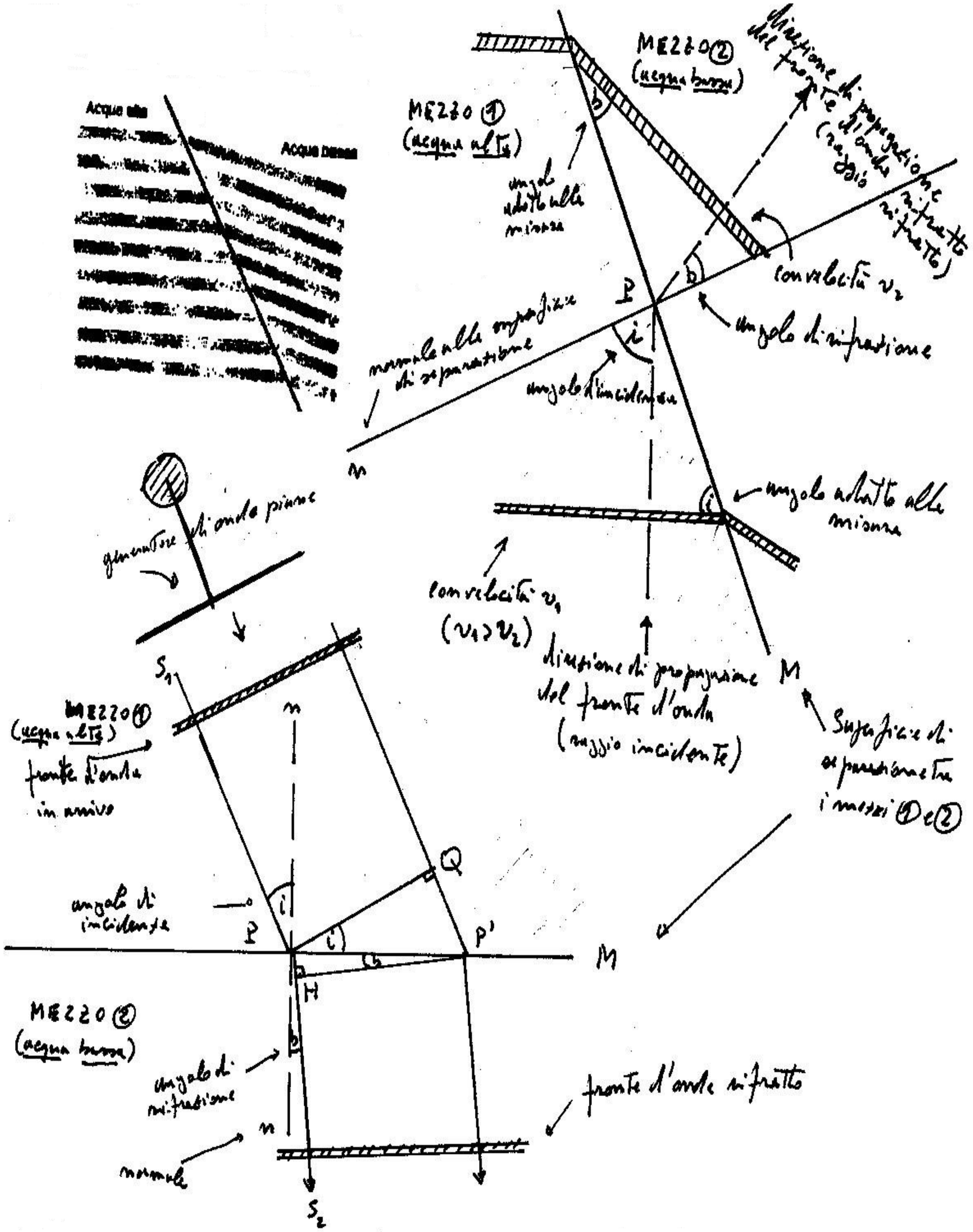
Assemble the machinery as showed in the picture, choosing a frequency between 15 Hz and 30 Hz. Wave's propagation speed in high water is superior than in shallow water, due to the fact that high water is thicker than shallow water. Indicate high water as *medium1* and shallow water as *medium 2*, high water velocity as v_1 and shallow water velocity as v_2. Given $v = \lambda f$, we have $v_1 = f\lambda_1$ and $v_2 = f\lambda_2$, where λ_1= wave length in high water and λ_2 = wave length in shallow water; *f*= chosen frequency.

Considering the speed of light = *c* , we can define the *absolute refractive index* of a medium: it is the relation $n = \frac{c}{v}$; in our situation we have $n_1 = \frac{c}{v_1}$ as absolute refractive index in high water and $n_2 = \frac{c}{v_2}$ as absolute refractive index in shallow water.

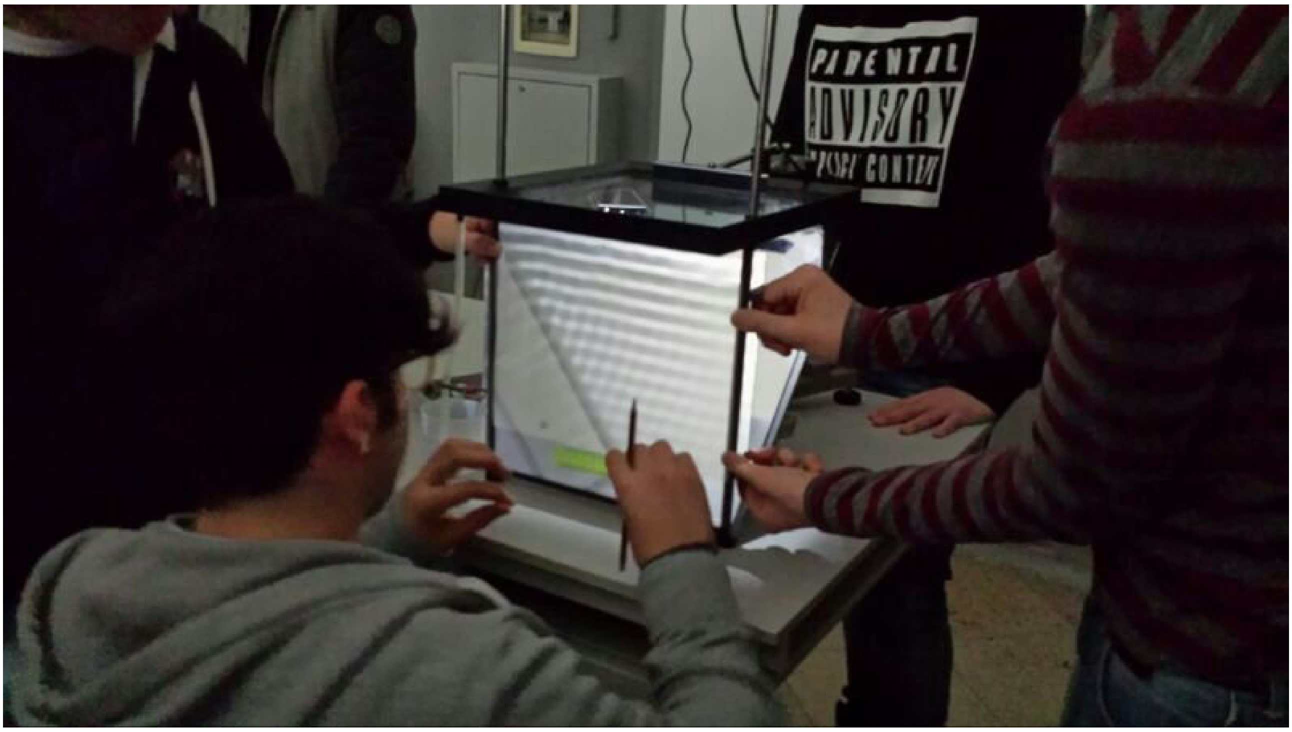
Being $v_1 > v_2$, it follows $\frac{c}{v_1} < \frac{c}{v_2}$ and so $n_1 < n_2$; in addition to this, we notice that the relation $\frac{v_1}{v_2} = \frac{\lambda_1}{\lambda_2}$ and $\frac{v_1}{v_2} = \frac{\frac{c}{n_1}}{\frac{c}{n_2}} = \frac{n_2}{n_1}$, therefore

$\frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} \geq n_{12} > 1$, the *refractive index related to the passage from medium 1 to medium 2*. Similarly, being $n_{21} > \frac{\lambda_2}{\lambda_1}$, it follows $n_{12} = \frac{1}{n_{21}}$.

Now, place a sheet of paper on the projection table and draw the limit between high water and shallow water (stressed by the plexiglass body); then draw 5 wave fronts both for high and shallow water. Calculate the wave length $\lambda_1$andλ_2 , then measure the angle of incidence *i* and the angle of refraction *b* using a goniometer; these angles can be also measured as the angles included between the wave front and the border of the body (see the picture).



Acqua alta
Acqua bassa
MEZZO ②
(acqua bassa)
MEZZO ①
(acqua alta)
direzione di propagazione del fronte d'onda (raggio rifratto)
angolo adatto alla misura
b
P
con velocità v_2
angolo di rifrazione
normale alla superficie di separazione
i
angolo d'incidenza
n
angolo adatto alla misura
con velocità v_1
($v_1 > v_2$)
direzione di propagazione del fronte d'onda (raggio incidente)
M
Superficie di separazione tra i mezzi ① e ②
generatore di onde piane
S_1
MEZZO ①
(acqua alta)
fronte d'onda in arrivo
n
Q
angolo di incidenza
P
i
P'
M
H
MEZZO ②
(acqua bassa)
b
angolo di rifrazione
fronte d'onda rifratto
n
normale
S_2



In the picture you can see how a group of plane waves spreads with a speed $v_1$in medium 1, meeting the plexiglass obstacle's surface separating medium 1 from medium 2, and it spreads (changing its direction) with a speed $v_2$in medium 2. Called P^lH the perpendicular to s_2, we notice that the points P and Q belong to the same wave front, and for this reason they must be in phase with the corresponding vibrations. Therefore, in the points P^l and H, there is the same phase value if the stretches QP^l ad PH are covered in the same time. It means that, if $t_1 = \frac{QP'}{v_1}$and $t_2 = \frac{PH}{v_2}$, we have ***t_1=t_2***. Observing that $Q\hat{P}P^l \cong n\hat{P}s_1$(they are both complementary of the angle $n\hat{P}Q$)then $H\hat{P}P^l \cong n\hat{P}s_2$they have the sides ***n*** and ***s_1***respectively perpendicular to $PP^l$and P^lH, we have that $\overline{QP'} = \overline{PP'}\sin i$, $\overline{PH} = \overline{PP'}\sin b$.

So we have: $t_1 = t_2 \;\Rightarrow\; \frac{QP'}{v_1} = \frac{PH}{v_2} \Rightarrow\; \frac{\overline{PP'}\sin i}{v_1} = \frac{\overline{PP'}\sin b}{v_2} \;\Rightarrow\; \frac{\sin i}{\sin b} = \frac{v_1}{v_2}$,

and, as we wrote in page 1,

$$\frac{\sin i}{\sin b} = \frac{\lambda_1}{\lambda_2} = \frac{n_2}{n_1} = n_{12}$$

refractive index related to the passage from medium 1 to medium 2. The result

$$\boxed{\frac{\sin i}{\sin b} = n_{12}}$$

is known as **Snell's law**, stating that"the relation between the sine of the angle of incidence and the sine of the angle of refraction is constant, whatever is the value of the angle of incidence."

In the case we studied in this experience we have $v_1 > v_2 \Rightarrow \frac{\sin i}{\sin b} > 1 \Rightarrow$ $\sin i > \sin b \Rightarrow i > b$ with $0 < i < \frac{\pi}{2}$, and the refracted rays get closer to the normal line ***n*** and to the separation surface ***M***, while in the opposite situation the rays distance from the normal line. When it happens, the refracted rays are parallel to the separation surface (for a certain limit angle of incidence), and for values superior than the angle of incidence there is no refracted ray, because there is a total refraction.

Of course, it is conceivable that the wave front begins from point Q in medium 2 and gets to point P in medium 1 with the same angles, with the only difference that ***b*** is the angle of incidence and ***i*** the angle of refraction. Snell's law will become:

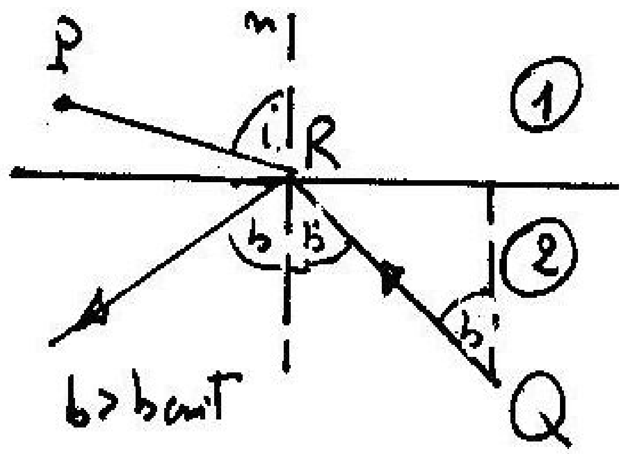
$$\frac{\sin b}{\sin i} = \frac{v_2}{v_1} = \frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2} = n_{21} < 1 \Rightarrow \sin b = n_{21} \sin i$$

Where, assumed $0 < i < \frac{\pi}{2} \Rightarrow 0 \leq \sin i \leq 1$, $\sin b \leq n_{21} \sin i$, we indicate that the sine of the angle of refraction does not overtake the refraction index. So, there must be a critic value of the angle of refraction that gives $\sin b_{crit} = n_{21} > 1$, which is the extreme value that the angle of refraction can have; with values $b > b_{crit}$ there is total refraction, where an observant from point Q, watching in the direction made by the perpendicular in Q with the angle of incidence $b > b_{crit}$ cannot see any wave front refracted by the separation surface, because he will see only wave fronts.

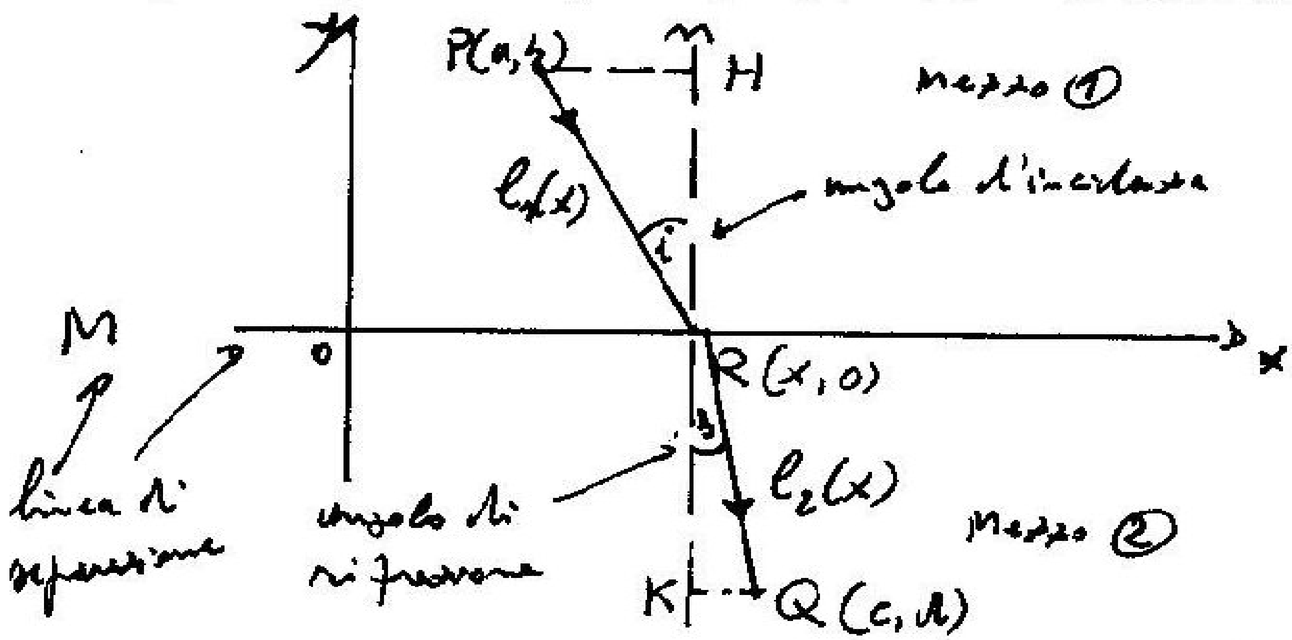
The incident ray distances from the normal line ***n***, making an angle ***i>b***. Therefore, $i = \frac{\pi}{2} \Rightarrow \sin i = 1$and, because $\sin b = n_{21}$, we have

$$\sin i = n_{21} b_{crit} = \arcsin n_{21}.$$

Let's go back to the case of the wave front passing from medium 1 to medium 2. Among all the possible paths, it will choose the shorter one; we can say it is lazy! (Fermat principle).



Taken a Cartesian system Oxy in the plane, we have:



The time needed to go from P to R is $t_1 = \frac{\overline{PR}}{v_1} = \frac{l_1(x)}{v_1}$ and the time needed to go from R to Q is $t_2 = \frac{\overline{RQ}}{v_2} = \frac{l_2(x)}{v_2}$. The overall time will be $t(x) > 0$,

$$t(x) = t_1 + t_2 = \frac{l_1(x)}{v_1} + \frac{l_2(x)}{v_2} = \frac{\sqrt{(x-a)^2+b^2}}{v_1} + \frac{\sqrt{(c-x)^2+d^2}}{v_2}.$$

So the Cartesian function in $\mathbb{R}$ $t(x) = \frac{\sqrt{(x-a)^2+b^2}}{v_1} + \frac{\sqrt{(c-x)^2+d^2}}{v_2}$ is derivable, positively divergent for divergent xs. Since its first derivative is

$$t'(x) = \frac{x-a}{v_1 l_1(x)} - \frac{c-x}{v_2 l_2(x)}$$

, if we solve the equation $t^I(x) = 0$ we will have $\frac{x-a}{v_1 l_1(x)} = \frac{c-x}{v_2 l_2(x)}$.

The x solution to that equation is $t(x)$'s minimum value, because if we study the sign of the derivative it results decreasing on the left and increasing on the right.

Observing that $\begin{cases} PH = x - a \\ QK = c - x \end{cases} \Rightarrow \begin{cases} PH = PR \sin i \\ QK = QR \sin b \end{cases} \Rightarrow \begin{cases} x - a = l_1(\mathrm{x}) \sin i \\ c - x = l_2(\mathrm{x}) \sin b \end{cases}$

Now, if we change the previous equation, we have:

$$\frac{l_1(x)\sin i}{v_1 l_1(x)} = \frac{l_2(x)\sin b}{v_2 l_2(x)} \quad \Rightarrow$$

$$\boxed{\frac{\sin i}{\sin b} = \frac{v_1}{v_2}}$$ **Snell's law.**

We still have to demonstrate that the solution to the equation

$$\frac{x-a}{v_1 l_1(x)} = \frac{c-x}{v_2 l_2(x)}$$

belongs to the interval [a, c]. So $x \in [a, c]$ **and Snell's law will be valid.**

When $x = a \Rightarrow H \equiv R \Rightarrow \frac{\sin i}{\sin b} = \frac{\frac{x-a}{l_1(x)}}{\frac{c-x}{l_2(x)}} \Rightarrow \frac{\sin i}{\sin b} = 0$

so the relation is null when $x = a$**.**

For $x \in (a, c)$**:**

$$\begin{cases} \sin i = \dfrac{x-a}{l_1(x)} = \dfrac{x-a}{\sqrt{(x-a)^2 + b^2}} = \dfrac{1}{\sqrt{1 + \frac{b^2}{(x-a)^2}}} \\ \sin b = \dfrac{c-x}{l_2(x)} = \dfrac{c-x}{\sqrt{(c-x)^2 + d^2}} = \dfrac{1}{\sqrt{1 + \frac{d^2}{(c-x)^2}}} \end{cases}$$

We can see that $\sin i$ **is a continue, positive, strictly increasing function in [a, c], while** $\sin b$ **is a continue, positive, strictly decreasing function in [a, c]; in addition to this,** $\lim_{x\to c} \frac{\sin i}{\sin b} = +\infty$ **, since:**

$$\lim_{x\to c} \sin i = \frac{1}{\sqrt{1+\frac{b^2}{(x-a)^2}}} \quad , \ \lim_{x\to c} b = \lim_{x\to c} \frac{1}{\sqrt{1+\frac{d^2}{(c-x)^2}}} = 0$$

It proves that the relation $\frac{\sin i}{\sin b}$ **reaches all the extreme values only once, so there must be an absolute minimum value.**